A CLASS OF CHANGE-OVER DESIGNS BALANCED FOR FIRST RESIDUAL EFFECTS

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1. Introduction

Designs in which each experimental unit receives a cyclical sequence of several treatments in successive periods are known as change-over designs. These designs are frequently used in several fields, notably in nutrition experiments with dairy cattle, in clinical trials in medical research, in psychological experiments and in bio-assays.

In experiments where the treatments are applied in sequence to the same experimental unit, sometimes the effect of treatments continues even after the application of the treatment is discontinued. That is, the effect of a treatment is influenced by the carry-over or *residual effect* of the previous treatment. Residual effects which persist only for one period are called *first order* residual effects, or, simply *first residual* effects. In this paper, we consider only first residual effects.

By suitably designing the experiment, it is possible to estimate the direct and residual effects of the treatments. A change-over design, permitting the estimation of residual effects, is called *balanced*, if the variance of any elementary contrast among the direct effect estimates is constant, say, α, and the variance of any elementary conrast among the estimated residual effects is also constant, say, β. The constants α and β need not be equal. However, if $\alpha=\beta$, the design may be called *totally balanced.*

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In this paper we consider change-over designs in which the number of periods (k) is smaller than the number of treatments (v). Change-over designs, balanced for first residual effects and with $k<v$ have been constructed by Patterson [2], Patterson and Lucas [4] and Saha [5]. In this paper, a new class of balanced change-over designs is constructed. These designs are totally balanced in the above sense and have been constructed using a special series of Balanced Incomplete Block (BIB) designs. The analysis of the proposed designs is discussed and the efficiency factors worked out. A table of designs alongwith the efficiency factors is also provided. The proposed designs exist for any v which is a prime or a prime-power of the form $mk+1$ and involve k periods and mv sequences. The efficiency factors of these designs are seen to be fairly high.

2. Balanced Change-over Designs

We denote the parameters of a BIB design by the usual symbols v, b, r, k, λ. Sprott[6] has shown that if $v(=mk+1)$ be a prime or a prime-power then the following series of BIB designs can be constructed :

$$v=mk+1,\ b=mv,\ r=mk,\ k,\ \lambda=k-1. \qquad ...(2.1)$$

The design (2.1) can be constructed by developing mod v, each of the following m initial blocks :

$$(x^i, x^{i+m}, x^{i+2m}, \ldots, x^{i+(k-1)m}), \qquad ...(2.2)$$

$$i=0, 1\ldots, m-1,$$

x being a primitive element of GF(v)

We shall use of the series (2.1) of BIB designs to construct balanced change-over designs.

If the blocks of the above BIB design are written as columns, we get a change-over design in v treatments, mv sequences and k periods, where the columns are treated as sequences and rows as periods. This change-over design is not balanced in our sense. We now modify the design slightly as described below :

The last treatment in each sequence is also placed in the same sequence in a period preceding the first period. Thus, from the BIB design (2.1) we get an arrangement of $k+1$ rows (periods) and mv columns (sequences) where the columns of the arrangement are the blocks of the BIB design (2.1), given by (2.2) and each column has the first and last treatments identical. We call the first row of this arrangement the *initial* period. The treatments are applied from the initial period prior to which the experiment really starts, though the observations in this period are not to be used for analysis. Actually, only the residual effects of the treatments applied in the initial period enter into the data collected from the second period.

In the next section we show that the modified design is a totally balanced change-over design.

As an example of the design, let $m=2$, $k=3$, so that $v=7$. The change-over design in 14 sequences and three periods is shown below :

Sequences

		I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
	O	4	5	6	7	1	2	3	5	6	7	1	2	3	4
	1	1	2	3	4	5	6	7	3	4	5	6	7	1	2
Periods	2	2	3	4	5	6	7	1	6	7	1	2	3	4	5
	3	4	5	6	7	1	2	3	5	6	7	1	2	3	4

The period numbered O is the initial period.

A Remark. Lawless[1] remarked that many BIB designs can be converted to change-over designs 'balanced' for first residual effects by putting the last treatment in every sequence (block of the BIB design) also in the begining of the sequence. By doing this, every treatment can be made to precede every other treatment equally frequently. Though such a property in any change-over design considerably facilitates the analysis, it does not necessarily guarantee the property of balance in the sense we are using. It appears that Lawless used the term 'balance' in the sense that every treatment is preceded by every other treatment equally frequently and not in the sense of equality of variances of the estimated elementary contrasts among the direct and residual effects.

3. Analysis and Efficiency Factors

Consider the design described in section 2 in $v(=mk+1)$ treatments, k periods and mv sequences, $m \geqslant 1, k \geqslant 3$. It can be seen easily that the design in k periods (excluding the initial period) satisfies the following properties :

I. No treatment occurs in a given sequence more than once;

II. every treatment occurs in a given period an equal number of times (precisely m times) ;

III. every pair of treatments occurs together in $(k-1)$ sequences ;

IV. in those sequènces in which a given treatment appears, all other treatments occur equally frequently (precisely $k-1$ times).

Further, when the initial period is also considered, we find that in the design with $k+1$ periods, every treatment is preceded and followed by every other treatment equally often (precisely once).

For the analysis of the proposed designs, we consider the usual homoscedastic fixed effects model,

$$y_{ij\lambda m} = \mu + \pi_i + a_j + \tau_\lambda + \rho_m + \epsilon_{ij\lambda m}$$

where $y_{ij\lambda m}$ is the observation corresponding to a treatment λ, preceded by a treatment by m, given to j-th sequence in the i-th period, μ is the general mean effect, π_i, the effect of the period i, a_j, the j-th sequence effect, τ_λ , the direct effect of treatment; $\epsilon_{ij\lambda m}$'s are random error components, assumed to be independently and normally distributed about a zero mean with a constant variance, σ^2 ; $i=1, \ldots, k$, $j=1, \ldots, mv$, $\lambda, m=1, \ldots, v$, $\lambda \neq m$. Under the restrictions

$$\sum_i \pi_i = 0 = \sum_j a_j = \sum_\lambda \tau_\lambda = \sum_m \rho_m,$$

we get the normal equations for estimating the direct and residual effects for the designs under cosideration as :

$$T_i = mk\mu + \sum_\lambda a_\lambda^{(i)} + mk\tau_i - \rho_i, \quad (i=1, 2, \ldots, v) \qquad \ldots(3.1)$$

$$R_i = mk\mu + \sum_\lambda a_\lambda^{(i)} - \tau_i + mk\rho_i, \qquad \ldots(3.2)$$

where T_i is the total of observations which contain the direct effect of the i-th treatment, R_i, the total of observations receiving the residual effect of the i-th treatment, $\sum_\lambda a_\lambda^{(i)}$ denotes the total of those sequence effects which contain the treatment i.

Let $\sum_\lambda S_\lambda^{(i)}$ denote the total of observations of those sequences which contain the i-th treatment. Then, we have the following normal equations :

$$\sum_\lambda S_\lambda^{(i)} = mk^2\mu + k\sum_\lambda a_\lambda^{(i)} + mk\tau_i - (k-1)\tau_i + mk\rho_i - (k-1)\rho_i$$

or,

$$\sum_\lambda S_\lambda^{(i)} = mk^2\mu + k\sum_\lambda a_\lambda^{(i)} + (mk-k+1)(\tau_i+\rho_i), \qquad \ldots(3.3)$$

$$(i=1, 2, \ldots v).$$

Eliminating $\sum_\lambda a_\lambda^{(i)}$ from (3.1) and (3.2) we obtain the following equations in τ_i and ρ_i :

$$kP_i = (mk^2 - mk + k - 1)\tau_i - (mk+1)\rho_i, \qquad \ldots(3.4)$$

$$kQ_i = -(mk+1)\tau_i + (mk^2 - mk + k - 1)\rho_i, \qquad \ldots(3.5)$$

where $P_i = T_i - k^{-1}\sum_\lambda S\lambda^{(i)}, Q_i = R_i - k^{-1}\sum_\lambda S\lambda^{(i)}$.

Thus,

$$\check{\tau}_i = [Q_i + (k-1)P_i]/[(mk+1)(k-2)],$$

$$\hat{\rho}_i = [P_i + (k-1)Q_i]/[(mk+1)(k-2)].$$

The variance of any elementary contrast between the direct effect estimates is given by

$$V(\hat{\tau}_i - \hat{\tau}_j) = 2(k-1)\sigma^2/[mk+1)(k-2)] \qquad \ldots(3.6)$$

and that of any elementary contrast between the estimated residual effects is

$$V(\hat{\rho}_i - \hat{\rho}_j) = 2(k-1)\sigma^2/[(mk+1)(k-2)]. \qquad \ldots(3.7)$$

Since

$$V(\hat{\tau}_i-\hat{\tau}_j)=V(\hat{\rho}_i-\hat{\rho}_j) \text{ for all } i\neq j,$$

these designs are totally balanced.

If the *permanent effect* of the i-th treatment is defined as

$$e_i=\tau_i+\rho_i, \text{ we have}$$

$$V(\hat{e}_i-\hat{e}_j)=4k\sigma^2/[mk+1)(k-2)]. \qquad \ldots(3.8)$$

If the residual effects turn out to be non significant, we may estimate the direct effects under the hypothesis of no residual effects. Let the direct effect of the i-th treatment ignoring residual effects be $\overset{*}{\tau}_i$. The normal equation for estimating $\overset{*}{\tau}_i$ is

$$kP_i=(mk^2-mk+k-1)\overset{*}{\tau}_j.$$

Thus,

$$V(\hat{\overset{*}{\tau}}_i-\hat{\overset{*}{\tau}}_j)=2k\sigma^2/[(mk+1)(k-1)]. \qquad \ldots(3.9)$$

Let E_d, E_r, E_p, E_t respectively denote the efficiency factors for direct, residual, permanent and direct ignoring residual effects. These efficiency factors have been defined by Patterson and Lucas[4] as follows :

$$E_d=(2\sigma^2/r)/V(\hat{\tau}_i-\hat{\tau}_j),\ E_r=(2\sigma^2/r)/V(\hat{\rho}_i-\hat{\rho}_j),$$

$$E_p=(2\sigma^2/r)/V(\hat{e}_i-\hat{e}_j),\ E_t=(2\sigma^2/r)V(\hat{\tau}_i-\hat{\tau}_j),$$

where r is the number of times any treatment has appeared in the design. In the case of the designs proposed, $r=mk$ and thus the efficiency factors are given by

$$E_d=[(k-2)(mk+1)]/[mk(k-1)]=E_r,$$
$$E_p=(mk+1)(k-2)/2mk^2,$$
$$E_t=(mk+1)(k-1)]/mk^2 \qquad \ldots(3.10)$$

These efficiency factors (expressed as percentages) have been presented in a table for all permissible values of v in the range $7\leq v\leq 31$ and $3\leq k\leq 12$, requiring 150 units or less. The solutions of the designs are not reported, as these can be built up easily from the preceding discussions.

TABLE

Index of balanced change-over designs and percentage efficiency factors.

(1) S.No.	(2) v	(3) k	(4) n^*	(5) $E_d=E_r$	(6) E_p	(7) E_t
1	4	3	4	67	22	89
2	5	4	5	83	31	94
3	7	3	14	58	19	78
4	7	6	7	93	39	97
5	8	7	8	95	41	98
6	9	4	18	75	28	84
7	9	8	9	96	42	98
8	11	5	22	83	33	88
9	11	10	11	98	44	99
10	13	3	52	54	18	72
11	13	4	39	72	28	81
12	13	6	26	87	36	90
13	13	12	13	98	45	99
14	16	3	80	53	18	71
15	16	5	48	80	32	85
16	17	4	68	71	27	80
17	17	8	34	91	40	93
18	19	3	114	53	18	70
19	19	6	57	84	35	88
20	19	9	38	92	41	94
21	23	11	46	94	43	95
22	25	4	150	69	26	78
23	25	6	100	83	35	87
24	25	8	75	89	39	91
25	25	12	50	95	43	95
26	29	7	116	86	37	89
27	31	10	93	92	41	93

*n in the above table denotes the total number of sequences (units) in the design.

Summary

Using a series of *BIB* design, class of change-over design is constructed. These designs are balanced for first residual effects. A list of useful designs along with their efficiency factors is also provided.

References

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